Geometric phases of modulated multivariate time series

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Since their discovery by Sir M.V. Berry in 1983 [1], *Geometric Phases* (GPs) have been found to be present in many classical and quantum systems, and to be at the root of some exotic properties for these systems. Examples include the Foucault pendulum, the quantum Hall effect, the Aharonov–Bohm effect, and —more anecdotally— falling cats, to name a few [2]. Research fields such as topological quantum materials, condensed matter physics and quantum information make heavy use of those phases for theoretical purposes or experimental setups [3, 4]. The concept of geometric phase is related to the notion of path in a principal fibre bundle. In quantum mechanics, it is actually a manifestation of the path followed by the *projector* (associated to a quantum state) in the parameter space of the Hamiltonian of the system. After completing the path in the parameter space, the state of the system is affected by an additional phase term (the geometric phase) which is only proportional to the area surrounded by the path.

In this work, we show that two types of GPs may appear when analyzing multivariate time series: a *commutative* geometric phase of Pancharatnam-Berry type and a *non-commutative* one. The occurence of these phases is due to the structure of the space when considering complex valued *modulated* multivariate signals. Such signals $\boldsymbol{x}(t) \in \mathbb{C}^n \setminus \{0\}$, with $n \geq 2$, may be represented in the following way $\boldsymbol{x}(t) = a(t)e^{\boldsymbol{i}\varphi(t)}\boldsymbol{p}(t)$, where $a(t) \in \mathbb{R}_+$ is the amplitude, $e^{\boldsymbol{i}\varphi(t)} \in \mathbf{U}(1)$ a phasor (a phase term shared by all the components of $\boldsymbol{x}(t)$ that allows to define the concept of an instantaneous frequency for the signal) and $\boldsymbol{p}(t) \in \mathcal{S}^{2n-1}$ a normalized complex vector which contains the instantaneous correlations between the components of $\boldsymbol{x}(t)$. Using such a model consists in imposing a fiber bundle structure for the space of (normalized) modulated multivariate signals, with $\mathcal{S}^{2n-1}$ as the total space, $\mathrm{P}\mathbb{C}^{n-1}$ the base space (with elements $\boldsymbol{p}(t)\boldsymbol{p}(t)^\dagger$) and $\mathbf{U}(1)$ the fibre (and structural group). The geometric phase is actually directly given by integrating the connection in this space along the path followed by the signal during a time lapse. Equipped with this geometric structure, we show that two kinds of geometric phases can affect the interpretation of some observables classically computed in signal processing :

1. Commutative GP: when computing the *instantaneous frequency* of a multivariate signal, a geometric phase term can potentially corrupt the estimated frequency.
2. Non-Commutative GP: when the vector $\boldsymbol{p}(t)$ is of dimension $k < n$ (with $n \geq 3$), a non-commutative geometric phase term (an element of $\mathbf{U}(k)$) may "*rotates*" the subspace spanned by the signal (this subspace is an element of the k-dimensional Stiefel manifold in $\mathbb{C}^n$).

The conditions for these effects to appear will be detailed (geodesic path, open or closed loop), together with the properties of the GPs (gauge and time reparametrization invariance) and their significance for multivariate signals. In terms of computation, we provide a recursive estimator for the estimation of the geometric phase for discretized signals with better robustness and efficiency than "standard" estimators based on Bargmann invariant [5]. As an illustration, we will provide simulations highlighting the possible usage of commutative GP for *polarization state fluctuations* and non-commutative GP for *polarization plane fluctuations* in the case of polarized gravitational wave analysis for the characterization of precessing compact binary merger.

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References

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